

Low-lying hypernuclei in the relativistic quark-gluon model

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Low-lying hypernuclei ${}^3_{\Lambda}H$, ${}^3_{\Sigma}H$, ${}^3_{\Lambda}He$, ${}^3_{\Sigma}He$ are described by the relativistic nine-quark equations in the framework of the dispersion relation technique. The approximate solutions of these equations using the method based on the extraction of leading singularities of the amplitudes are obtained. The relativistic nine-quark amplitudes of hypernuclei, including the quarks of three flavors (u , d , s) are calculated. The poles of these amplitudes determine the masses of hypernuclei. The mass of state ${}^3_{\Lambda}H$ with the isospin $I = 0$ and the spin-parity $J^P = \frac{1}{2}^+$ is equal to $M = 2952 \text{ MeV}$.

PACS numbers: 11.55.Fv, 12.39.Ki, 12.39.Mk.

I. INTRODUCTION.

Hypernuclei spectroscopy is enjoying an experimental renaissance with ongoing and planned program at DAΦNE, FAIR, Jefferson Lab, J-PARC and Mainz providing motivation for enhanced theoretical efforts (for a recent review, see Ref. [1]).

LQCD calculations [2–6] are performed using an isotropic quark action at the $SU(3)$ flavor symmetric point corresponding to the physical strange quark mass, with $m_{\pi} = m_K = m_{\eta} \sim 800 \text{ MeV}$. The lattice spacing $b = 0.145 \text{ fm}$ has been used in the calculations, dictated by the available computational resources, and therefore an extrapolation to the continuum has not been performed. Further, extrapolation of the physical pion mass have not been attempted, because the quark mass dependence of the energy levels in the light nuclei are not known. Future calculations at smaller lattice spacing and at lighter quark masses will facilitate such extrapolations and lead to first predictions for the spectrum of light nuclei, with completely quantified uncertainties, that can be compared with experiment.

In the recent paper [7] the relativistic six-quark equations are found in the framework of coupled-channel formalism. The dynamical mixing between the subamplitudes of hexaquark are considered. The six-quark amplitudes of dibaryons are calculated. The poles of these amplitudes determine the masses of dibaryons. We calculated the contribution of six-quark subamplitudes to the hexaquark amplitudes.

In the previous paper [8] the 3He as the system of interacting quarks and gluons is considered. The relativistic nine-quark equations are found in the framework of the dispersion relation technique. The dynamical mixing between the subamplitudes of 3He is taken into account. The relativistic nine-quark amplitudes of 3He , including the u , d quarks are calculated. The approximate solutions of these equations using the method based on the extraction of leading singularities of the amplitudes were obtained. The poles of the nonaquark amplitudes determined the mass of 3He . The mass of 3He state is equal to $M = 2809 \text{ MeV}$. The model use only two parameters: the cutoff $\Lambda = 11$ [9, 10] and gluon coupling constant $g = 0.1536$, which is smaller than the gluon coupling constant of baryon physics.

In the present paper the hypernuclei ${}^3_{\Lambda}H$, ${}^3_{\Sigma}H$, ${}^3_{\Lambda}He$, ${}^3_{\Sigma}He$, $nn\Lambda$ and $nn\Sigma$ are considered in the framework of the dispersion relation technique. The approximate solutions of the nine-quark equations using the method based on the extraction of leading singularities of the amplitude are obtained. The relativistic nona-amplitudes of low-lying hypernuclei, including the three flavors (u , d , s) are calculated. The poles of these amplitudes determine the masses of the hypernuclei. In Sec. II the relativistic nine-quark amplitudes of the hypernuclei are constructed. The dynamical mixing between the subamplitudes of hypernuclei are considered.

Sec. III is devoted to the calculation results for the masses of the lowest hypernuclei (Table I). In conclusion, the status of the considered model is discussed.

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TABLE I: Masses of lowest hypernuclei.

I	J^P	hypernuclei	quark content	mass
0	$\frac{1}{2}^+, \frac{3}{2}^+$	${}^3_{\Lambda}H$	$uud\,udd\,uds$	2952
1	$\frac{1}{2}^+, \frac{3}{2}^+$	${}^3_{\Sigma}H$	$uud\,udd\,uus$ ($uud\,udd\,dds$)	2956
0	$\frac{1}{2}^+, \frac{3}{2}^+$	${}^3_{\Sigma}He$	$uud\,uud\,dds$	2980
1	$\frac{1}{2}^+, \frac{3}{2}^+$	${}^3_{\Lambda}He$	$uud\,uud\,uds$	2953
2	$\frac{1}{2}^+, \frac{3}{2}^+$	${}^3_{\Sigma}He$	$uud\,uud\,uus$	2918

II. NINE-QUARK AMPLITUDES OF HYPERNUCLEI.

We derive the relativistic nine-quark equations in the framework of the dispersion relation technique. We use only planar diagrams; the other diagrams due to the rules of $1/N_c$ expansion [11–13] are neglected.

The current generates a nine-quark system. The correct equations for the amplitude are obtained by taking into account all possible subamplitudes. Then one should represent a nine-particle amplitude as a sum of 36 subamplitudes:

$$A = \sum_{\substack{i < j \\ i, j=1}}^9 A_{ij}. \quad (1)$$

This defines the division of the diagrams into groups according to the certain pair interaction of particles. The total amplitude can be represented graphically as a sum of diagrams. We need to consider only one group of diagrams and the amplitude corresponding to them, for example A_{12} . We consider the derivation of the relativistic generalization of the Faddeev-Yakubovsky approach.

In our case the two states of hypertritons are obtained: with the isospin $I = 0$ ${}^3_{\Lambda}H$ ($uud\,udd\,uds$) and with the isospin $I = 1$ ${}^3_{\Sigma}H$ ($uud\,udd\,uus$). The similar method allows us to construct the hyperhelium states: with the isospin $I = 0$ ${}^3_{\Sigma}He$ ($uud\,uud\,dds$), with the isospin $I = 1$ ${}^3_{\Lambda}He$ ($uud\,uud\,uds$) and with the isospin $I = 2$ ${}^3_{\Sigma}He$ ($uud\,uud\,uus$).

We take into account the pairwise interaction of all nine quarks. The set of diagrams associated with the amplitude A_{12} can further be broken down into some groups corresponding to subamplitudes similar to [8], Appendix A.

In order to represent the subamplitudes A_i in the form of a dispersion relation, it is necessary to define the amplitude of qq interactions. We use the results of our relativistic quark model [14]:

$$a_n(s_{ik}) = \frac{G_n^2(s_{ik})}{1 - B_n(s_{ik})}, \quad (2)$$

$$B_n(s_{ik}) = \int_{(m_i+m_k)^2}^{\Lambda \frac{(m_i+m_k)^2}{4}} \frac{ds'_{ik}}{\pi} \frac{\rho_n(s'_{ik}) G_n^2(s'_{ik})}{s'_{ik} - s_{ik}}. \quad (3)$$

Here $G_n(s_{ik})$ are the diquark vertex functions (Table II). The vertex functions are determined by the contribution of the crossing channels. The vertex functions satisfy the Fierz relations. These vertex functions are generated from g_V . $B_n(s_{ik})$ and $\rho_n(s_{ik})$ are the Chew-Mandelstam functions with cutoff Λ and the phase spaces:

$$\begin{aligned} \rho_n(s_{ik}, J^P) &= \left(\alpha(n, J^P) \frac{s_{ik}}{(m_i + m_k)^2} + \beta(n, J^P) + \delta(n, J^P) \frac{(m_i - m_k)^2}{s_{ik}} \right) \\ &\times \frac{\sqrt{(s_{ik} - (m_i + m_k)^2)(s_{ik} - (m_i - m_k)^2)}}{s_{ik}}. \end{aligned} \quad (4)$$

The coefficients $\alpha(n, J^P)$, $\beta(n, J^P)$ and $\delta(n, J^P)$ are given in Table II.

TABLE II: The vertex functions and coefficients of Chew-Mandelstam functions.

n	J^P	$G_n^2(s_{kl})$	α_n	β_n	δ_n
1	0^+	$\frac{4g}{3} - \frac{8gm_{kl}^2}{(3s_{kl})}$	$\frac{1}{2}$	$-\frac{1}{2} \frac{(m_k - m_l)^2}{(m_k + m_l)^2}$	0
2	1^+	$\frac{2g}{3}$	$\frac{1}{3}$	$\frac{4m_k m_l}{3(m_k + m_l)^2} - \frac{1}{6}$	$-\frac{1}{6}$

Here $n = 1$ corresponds to qq -pairs with $J^P = 0^+$, $n = 2$ corresponds to the qq pairs with $J^P = 1^+$. s_{ik} is the two-particle subenergy squared.

In the case in question the interacting quarks do not produce a bound states, therefore the integration is carried out from the threshold $(m_i + m_k)^2$ to the cutoff Λ .

Let us extract singularities in the coupled equations and obtain the reduced amplitudes α_i .

The isospin $I = 0$ ${}^3_\Lambda H$.

We need to consider the system of 29 equations:

$$\alpha_1 : \quad 1^{uu}, 1^{dd}, 0^{ud}, 0^{us}, 0^{ds} \quad (5)$$

$$\alpha_2 : \quad 1^{uu}1^{uu}, 1^{uu}1^{dd}, 1^{uu}0^{ud}, 1^{uu}0^{us}, 1^{uu}0^{ds}, \\ 1^{dd}1^{dd}, 1^{dd}0^{ud}, 1^{dd}0^{us}, 1^{dd}0^{ds}, \\ 0^{ud}0^{ud}, 0^{ud}0^{us}, 0^{ud}0^{ds} \quad (6)$$

$$\alpha_3 : \quad 1^{uu}1^{dd}0^{ud}, 1^{uu}1^{dd}0^{us}, 1^{uu}1^{dd}0^{ds}, \\ 1^{uu}0^{ud}0^{ud}, 1^{uu}0^{ud}0^{us}, 1^{uu}0^{ud}0^{ds}, \\ 0^{ud}1^{dd}0^{ud}, 0^{ud}1^{dd}0^{us}, 0^{ud}1^{dd}0^{ds}, \\ 0^{ud}0^{ud}0^{ud}, 0^{ud}0^{ud}0^{us}, 0^{ud}0^{ud}0^{ds} \quad (7)$$

The isospin $I = 1$ ${}^3_\Sigma H$ (24 equations):

$$\alpha_1 : \quad 1^{uu}, 1^{dd}, 0^{ud}, 0^{us}, 0^{ds} \quad (8)$$

$$\alpha_2 : \quad 1^{uu}1^{uu}, 1^{uu}1^{dd}, 1^{uu}0^{ud}, 1^{uu}0^{us}, 1^{uu}0^{ds}, \\ 1^{dd}0^{ud}, 1^{dd}0^{us}, 1^{dd}0^{ds}, \\ 0^{ud}0^{ud}, 0^{ud}0^{us}, 0^{ud}0^{ds} \quad (9)$$

$$\alpha_3 : \quad 1^{uu}1^{dd}1^{uu}, 1^{uu}1^{dd}0^{us}, \\ 1^{uu}0^{ud}1^{uu}, 1^{uu}0^{ud}0^{us}, \\ 0^{ud}1^{dd}1^{uu}, 0^{ud}1^{dd}0^{us}, \\ 0^{ud}0^{ud}1^{uu}, 0^{ud}0^{ud}0^{us} \quad (10)$$

Here the α_1 are determined by the diquarks, the α_2 includes the two diquarks and the five quarks. α_3 corresponds to the $pp\Lambda$ ($pn\Lambda$) states.

The equations for the ${}^3_\Lambda He$, ${}^3_\Sigma He$ are considered.

The isospin $I = 0$.

We need to consider 23 equations.

$$\alpha_1 : \quad 1^{uu}, 1^{dd}, 0^{ud}, 0^{us}, 0^{ds} \quad (11)$$

$$\alpha_2 : \quad 1^{uu}1^{uu}, 1^{uu}1^{dd}, 1^{uu}0^{ud}, 1^{uu}0^{us}, 1^{uu}0^{ds}, \\ 1^{dd}1^{dd}, 1^{dd}0^{ud}, 1^{dd}0^{us}, 1^{dd}0^{ds}, \\ 0^{ud}0^{ud}, 0^{ud}0^{us}, 0^{ud}0^{ds} \quad (12)$$

$$\alpha_3 : \quad 1^{uu}1^{uu}1^{dd}, 1^{uu}1^{uu}0^{ds}, \\ 1^{uu}0^{ud}1^{dd}, 1^{uu}0^{ud}0^{ds}, \\ 0^{ud}0^{ud}1^{dd}, 0^{ud}0^{ud}0^{ds} \quad (13)$$

The isospin $I = 1$ (25 equations):

$$\alpha_1 : \quad 1^{uu}, 1^{dd}, 0^{ud}, 0^{us}, 0^{ds} \quad (14)$$

$$\alpha_2 : \quad 1^{uu}1^{uu}, 1^{uu}1^{dd}, 1^{uu}0^{ud}, 1^{uu}0^{us}, 1^{uu}0^{ds}, \\ 1^{dd}0^{ud}, 1^{dd}0^{us}, 1^{dd}0^{ds}, \\ 0^{ud}0^{ud}, 0^{ud}0^{us}, 0^{ud}0^{ds} \quad (15)$$

$$\alpha_3 : \quad 1^{uu}1^{uu}0^{ud}, 1^{uu}1^{uu}0^{us}, 1^{uu}1^{uu}0^{ds}, \\ 1^{uu}0^{ud}0^{ud}, 1^{uu}0^{ud}0^{us}, 1^{uu}0^{ud}0^{ds}, \\ 0^{ud}0^{ud}0^{ud}, 0^{ud}0^{ud}0^{us}, 0^{ud}0^{ud}0^{ds} \quad (16)$$

The isospin $I = 2$ (20 equations):

$$\alpha_1 : \quad 1^{uu}, 1^{dd}, 0^{ud}, 0^{us}, 0^{ds} \quad (17)$$

$$\alpha_2 : \quad 1^{uu}1^{uu}, 1^{uu}1^{dd}, 1^{uu}0^{ud}, 1^{uu}0^{us}, 1^{uu}0^{ds}, \\ 1^{dd}0^{us}, \\ 0^{ud}0^{ud}, 0^{ud}0^{us}, 0^{ud}0^{ds} \quad (18)$$

$$\alpha_3 : \quad 1^{uu}1^{uu}1^{uu}, 1^{uu}1^{uu}0^{us}, \\ 1^{uu}0^{ud}1^{uu}, 1^{uu}0^{ud}0^{us}, \\ 0^{ud}0^{ud}1^{uu}, 0^{ud}0^{ud}0^{us} \quad (19)$$

The coefficients of the coupled equations are determined by the permutation of quarks [15, 16]. For the simplicity we consider the graphical equation of the reduced amplitude $\alpha_3^{1^{uu}1^{uu}1^{uu}}$ (Fig. 1).

In Fig. 1 the first coefficient is equal to 12, that the number $12 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 3$ (permutation of pairs (56) and (12), (56) and (34)); the second coefficient equal to 12, that the number $12 = 2$ (permutation of particles 1 and 2) $\times 3$ (permutation of pairs (12) and (34), (12) and (56)) $\times 2$ (we can replace 7-th d -quark with 8-th d -quark); the third coefficient equal to 6, that the number $6 = 2$ (permutation of particles 1 and 2) $\times 3$ (permutation of pairs (12) and (34), (12) and (56)); the 4-th coefficient equal to 6: $6 = 2$ (permutation of d -quarks 7 and 8) $\times 3$ (permutation of pairs (12) and (34), (12) and (56)); the 5-th coefficient equal to 12: $12 = 2$ (permutation of d -quarks 7 and 9) $\times 2$ (permutation of d -quarks 7 and 8) $\times 3$ (permutation of pairs (12) and (34), (12) and (56)); the 6-th coefficient equal to 24: $24 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 3$ (permutation of pairs (56) and (12), (56) and (34)) $\times 2$ (permutation of d -quarks 7 and 8); the 7-th coefficient equal to 48: $48 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 3$ (permutation of pairs (56) and (12), (56) and (34)) $\times 2$ (permutation of d -quarks 7 and 9) $\times 2$ (permutation of d -quark 7 and s -quark 8); the 8-th coefficient equal to 48: $48 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 2$ (permutation of pairs (12) and (34)) $\times 3$ (permutation of pairs (56) and (12), (56) and (34)) $\times 2$ (permutation of d -quarks 7 and 8); the 9-th coefficient equal to 24: $24 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 2$ (permutation of pairs (12) and (34)) $\times 3$ (permutation of pairs (56) and (12), (56) and (34)); the 10-th coefficient equal to 24 (this case is similar to 6-th); the 11-th coefficient equal to 48 (this case is similar to 7-th); the 12-th coefficient equal to 48: $48 = 2$ (permutation of particles 1 and 2) $\times 6$ (permutation of pairs (12), (34) and (56)) $\times 2$ (permutation of d -quarks 7 and 8) $\times 2$ (permutation of d -quark 8 and s -quark 9); the 13-th coefficient equal to 24: $24 = 2$ (permutation of particles 1 and 2) $\times 6$ (permutation of pairs (12), (34) and (56)) $\times 2$ (permutation of d -quarks 8 and 9).

The similar approach allows us to take into account the coefficients in all equations.

The functions $I_1 - I_{10}$ are taken from the paper [7]. The other functions are determined by the following formulae [8] (see also Fig. 1):

$$I_{11}(12, 34, 15, 36, 47) = I_1(12, 15) \times I_2(34, 36, 47), \quad (20)$$

$$I_{12}(12, 34, 56, 17) = I_1(12, 17), \quad (21)$$

$$I_{13}(12, 34, 56, 17, 28) = I_2(12, 17, 28), \quad (22)$$

$$I_{14}(12, 34, 56, 17, 38) = I_1(12, 17) \times I_1(34, 38), \quad (23)$$

$$I_{15}(12, 34, 56, 23, 47) = I_7(12, 34, 23, 47), \quad (24)$$

$$I_{16}(12, 34, 56, 17, 23, 48) = I_8(12, 34, 17, 23, 48), \quad (25)$$

$$I_{17}(12, 34, 56, 17, 45) = I_1(12, 17) \times I_3(34, 56, 45), \quad (26)$$

$$I_{18}(12, 34, 56, 17, 38, 49) = I_1(12, 17) \times I_2(34, 38, 49), \quad (27)$$

$$I_{19}(12, 34, 56, 17, 38, 59) = I_1(12, 17) \times I_1(34, 38) \times I_1(56, 59), \quad (28)$$

$$I_{20}(12, 34, 56, 17, 45, 68) = I_1(12, 17) \times I_7(34, 56, 45, 68), \quad (29)$$

$$I_{21}(12, 34, 56, 17, 28, 45) = I_2(12, 17, 28) \times I_3(34, 56, 45). \quad (30)$$

The main contributions are determined by the functions I_1 and I_2 :

$$I_1(ij) = \frac{B_j(s_0^{13})}{B_i(s_0^{12})} \int_{(m_1+m_2)^2}^{\Lambda \frac{(m_1+m_2)^2}{4}} \frac{ds'_{12}}{\pi} \frac{G_i^2(s_0^{12}) \rho_i(s'_{12})}{s'_{12} - s_0^{12}} \int_{-1}^{+1} \frac{dz_1(1)}{2} \frac{1}{1 - B_j(s'_{13})}, \quad (31)$$

$$\begin{aligned} I_2(ijk) &= \frac{B_j(s_0^{13}) B_k(s_0^{24})}{B_i(s_0^{12})} \int_{(m_1+m_2)^2}^{\Lambda \frac{(m_1+m_2)^2}{4}} \frac{ds'_{12}}{\pi} \frac{G_i^2(s_0^{12}) \rho_i(s'_{12})}{s'_{12} - s_0^{12}} \frac{1}{2\pi} \int_{-1}^{+1} \frac{dz_1(2)}{2} \int_{-1}^{+1} \frac{dz_2(2)}{2} \\ &\times \int_{z_3(2)^-}^{z_3(2)^+} dz_3(2) \frac{1}{\sqrt{1 - z_1^2(2) - z_2^2(2) - z_3^2(2) + 2z_1(2)z_2(2)z_3(2)}} \\ &\times \frac{1}{1 - B_j(s'_{13})} \frac{1}{1 - B_k(s'_{24})}, \end{aligned} \quad (32)$$

where i, j, k correspond to the diquarks with the spin-parity $J^P = 0^+, 1^+$.

The one new contribution $I_{22} \sim 10^{-8}$. The contributions of $I_{10}, I_{17}, I_{19}, I_{20}, I_{21}$ also are small $\sim 10^{-7}$. We do not take into account these functions in the coupled equations.

III. CALCULATION RESULTS.

The poles of the reduced amplitudes correspond to the bound state and determine the masses of low-lying hypernuclei. The quark masses $m = 410 \text{ MeV}$ and $m_s = 557 \text{ MeV}$ coincide with the quark masses of the ordinary baryon in our model [9, 10]. This model does not have a new parameter as compared with the Ref. [8]. The gluon coupling constant $g = 0.1536$ is determined by fixing the mass of ${}^3\text{He}$ at $M = 2809 \text{ MeV}$. The cutoff parameter is similar to ordinary value $\Lambda = 11$. The calculated mass values of the hypernuclei are shown in the Table I.

We predict the degeneracy of hypertriton and hyperhelium with the spin-parity $J^P = \frac{1}{2}^+, \frac{3}{2}^+$.

The hypertriton with the $I = 0 \text{ } {}^3_\Lambda H$ and the hyperhelium with the $I = 1 \text{ } {}^3_\Lambda He$ also can be considered with the degeneracy.

IV. CONCLUSION.

It is now clear, but hardly a surprise, that the spectrum of nuclei and hypernuclei changes dramatically from light-quark masses. While one had already shown this from the recent work [17] on the H -dibaryon, and nucleon-nucleon scattering lengths, this has now been demonstrated to be true for even larger systems.

It will be interesting to learn how the various thresholds for binding evolve with the light quark masses providing accurate binding energies for any given light quark masses will require the inclusion of electromagnetic effect.

Lattice QCD has evolved to the point where first principles calculation of light nuclei are now possible, as demonstrated by the calculation at unphysically heavy quark masses presented in the paper [18]. The experimental program in hypernuclear physics, and the difficulties encountered in accurately determined rates for low energy nuclear reactions, warrant continued effort in, and development of, the application of LQCD to nuclear physics.

Clearly, calculation at smaller lattice spacings at $SU(3)$ symmetric point are required in order to remove the systematic uncertainties in the nuclear binding energies at these quark masses. In order to impact directly the experimental program in nuclear and hypernuclear physics, analogous calculations must be performed at lighter quark masses, ideally at or close to their physical values.

The antihypertriton have the mass that is similar to the hypertriton in our calculations.

The experimental data of hypertriton mass is about 3 GeV [19].

Acknowledgments

The authors would like to thank T. Barnes and L.V. Krasnov for useful discussions.

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$$\alpha_3^{1^{uu}1^{uu}1^{uu}} \quad \lambda \quad + \quad 12 I_9(1^{uu}1^{uu}1^{uu}1^{uu}) \alpha_1^{1^{uu}} \quad + \quad 12 I_{12}(1^{uu}1^{uu}1^{uu}0^{ud}) \alpha_1^{0^{ud}}$$

$$6 I_{12}(1^{uu}1^{uu}1^{uu}0^{us}) \alpha_1^{0^{us}} \quad + \quad 6 I_{13}(1^{uu}1^{uu}0^{ud}0^{ud}) \alpha_2^{0^{ud}0^{ud}} \quad + \quad 12 I_{13}(1^{uu}1^{uu}0^{ud}0^{us}) \alpha_2^{0^{ud}0^{us}}$$

$$24 I_{14}(1^{uu}1^{uu}0^{ud}0^{ud}) \alpha_2^{0^{ud}0^{ud}} \quad + \quad 48 I_{14}(1^{uu}1^{uu}0^{ud}0^{us}) \alpha_2^{0^{ud}0^{us}} \quad + \quad 48 I_{15}(1^{uu}1^{uu}1^{uu}0^{ud}) \alpha_2^{1^{uu}0^{ud}}$$

$$24 I_{15}(1^{uu}1^{uu}1^{uu}0^{us}) \alpha_2^{1^{uu}0^{us}} \quad + \quad 24 I_{16}(1^{uu}1^{uu}1^{uu}0^{ud}1^{uu}0^{ud}) \alpha_3^{0^{ud}0^{ud}1^{uu}} \quad + \quad 48 I_{16}(1^{uu}1^{uu}1^{uu}0^{ud}1^{uu}0^{us}) \alpha_3^{1^{uu}0^{ud}0^{us}}$$

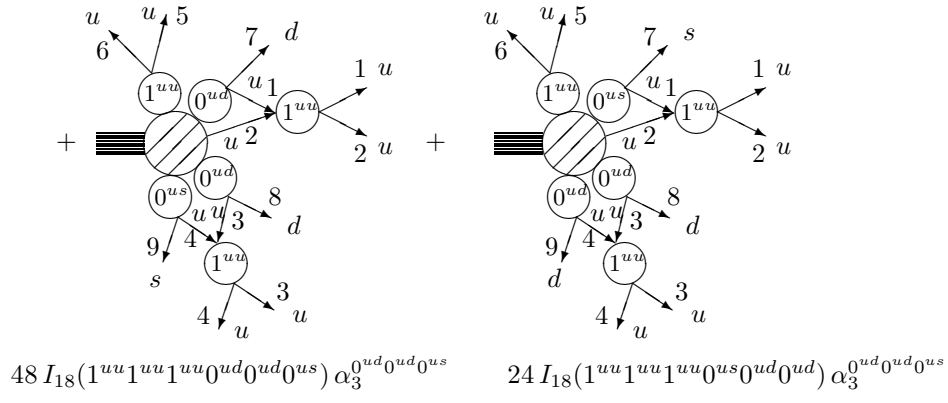


Fig. 1. The graphical equations of the reduced amplitude $\alpha_3^{1^{uu} 1^{uu} 1^{uu}} I = 2 \frac{3}{\Sigma} He$.